E- and M-functions: concordance and divergence

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E- and M-functions

Definition (Siegel 1929)

$f(z) \in \overline{\mathbb{Q}}[[z]]$ is an *E-function* if there exist $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0 f(z) + p_1 f'(z) + \cdots + p_m f^{(m)}(z) = 0$$

and the coefficients of $f(z)$ satisfy some arithmetic conditions.

Definition (inspired by Mahler 1929)

$f(z) \in \overline{\mathbb{Q}}[[z]]$ is an *M_q-function* if there exist $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0 f(z) + p_1 f(z^q) + \cdots + p_m f(z^{q^m}) = 0 .$$

The parameter $q \geq 2$ is an integer.

Examples. Important functions occurring in mathematics and physics: $\exp(z), \sin(z), \cos(z)$,

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2 2^{2n}} z^{2n},$$

and (some) hypergeometric functions.

Examples. Functions related to combinatorics and computer science:

$$\sum_{n=0}^{\infty} z^{q^n}, \prod_{n=0}^{\infty} \frac{1}{1 - z^{q^n}}, \sum_{n=0}^{\infty} s_q(n) z^n,$$

and the generating series of the sequences produced by finite automata.

Analytic beviour

An E-function:

- is an entire function.

An M-function:

- is analytic in some neighborhood of zero and meromorphic inside the open unit disc.
- is either rational or has the unit circle as a natural boundary (Randé, 1992).
- cannot satisfy an algebraic differential equation, unless it is rational (A., Dreyfus, and Hardouin 2021).

Remark

A $\star$-function is either rational or transcendental.

Motivation

The theory of E-functions is based on examples.

- The amazing results from the 19th century concerning the exponential function, leading to the transcendence of e, π, $\log 2$, and to the impossibility of squaring the circle.
- The results of Siegel concerning the Bessel functions.

The study of M-functions is mainly motivated by old problems concerning the complexity of integer base expansions of real numbers (É. Borel, Turing, Morse-Hedlund, Hartmanis-Stearns, Furstenberg...).

P1 What can be said about the complexity of the decimal expansion of $\sqrt{2}$, π, e or $\log 2$?

P2 Is it possible for a real irrational number to have a simple expansion both in base 2 and 3?

Achieving such goals requires the most general results of this theory!

Main motto

The algebraic relations over $\overline{\mathbb{Q}}$ between the values of $\star$-functions at algebraic points have a functional origin: they are governed by the algebraic relations over $\overline{\mathbb{Q}}(z)$ between these functions.

Linear systems and singularities

One studies linear systems of the form:

$$\boldsymbol{Y}'(z) = A(z)\boldsymbol{Y}(z)$$

with $A(z) \in \mathcal{M}_n(\overline{\mathbb{Q}}(z))$.

A point α is regular if the matrix $A(z)$ is well-defined at α.

One studies linear systems of the form:

$$\boldsymbol{Y}(z^q) = A(z)\boldsymbol{Y}(z)$$

with $A(z) \in \mathrm{GL}_n(\overline{\mathbb{Q}}(z))$.

A point α is regular if, for all $k \gg 1$, the matrix

$$A_k(z) := A(z^{q^{k-1}})A(z^{q^{k-2}}) \cdots A(z)$$

is well-defined and invertible at α.

Remark

An E-system has only finitely many singularities, while an M-system can have infinitely many singularities. However, it has only finitely many singularities on each compact subset of the open unit disc.

Quantitative result: Equality of transcendence degrees

Theorem

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be $\star$-functions that form the entries of a solution vector of a linear $\star$-system. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be a regular point. Then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), \ldots, f_m(\alpha)) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)) .$$

- First proof by Shidlovskii (1956) using Siegel's method.
- Second proof by André (2000) using the theory of E-operators.
- First proof by Ku. Nishioka (1990) using Nesterenko's approach.
- Second proof by A. & Faverjon (2020) using the pioneering ideas of Mahler.
- Proofs also work in p-adic settings and over $\mathbb{F}_q(t)$ (Fernandes, 2018).

Remark

The Galois theories associated with linear differential/difference equations provide tools to compute the value of $\mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z))$.

Consequences

Theorem

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be $\star$-functions that form the entries of a solution vector of a linear $\star$-system. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be a regular point. Then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), \ldots, f_m(\alpha)) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)) \,.$$

Equations of order 1.

- If $f(z)$ is a transcendental $\star$-function of order 1, then $f(\alpha)$ is transcendental if α is regular and $f(\alpha) = 0$ otherwise.

Maximal transcendence degree.

- If $\mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)) = m$, then $f_1(\alpha), \ldots, f_m(\alpha)$ are algebraically independent if α is regular and linearly dependent otherwise.

Generic behaviour.

- Let $f_1(z), \ldots, f_r(z)$ be algebraically independent $\star$-functions. Then they take algebraically independent values at almost all algebraic points.

Qualitative result

Lifting Theorem

Let $f_1(z), \ldots, f_m(z) \in \overline{\mathbb{Q}}[[z]]$ be $\star$-functions that form the entries of a solution vector of a linear $\star$-system. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be a regular point. Then for every homogeneous $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that $P(f_1(\alpha), \ldots, f_m(\alpha)) = 0$, there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_m]$, homogeneous in $X_1, \ldots, X_m$, such that

$$\begin{aligned} Q(z, f_1(z), \ldots, f_m(z)) &= 0 \\ Q(\alpha, X_1, \ldots, X_m) &= P(X_1, \ldots, X_m)\,. \end{aligned}$$

- First proof by Beukers (2006) using the theory of E-operators.
- Second proof by André (2014) derived from the quantitative statement.
- First proof by Philippon (2015) (and A. & Faverjon, 2017) derived from the quantitative statement.
- Second proof by Nagy and Szamuely (2020) “à la André".
- Third proof by A. & Faverjon (2020) using the pioneering ideas of Mahler.

Consequence

Theorem

Given a $\star$-function $f(z)$ and an algebraic point α, there exists an algorithm that determines whether $f(\alpha)$ is algebraic or transcendental.

- A. & Rivoal (2018)
- A. & Faverjon (2018)

Qualitative result II: algebro-differential and σ-algebraic relations

A $\frac{d}{dz}$-algebraic relation between $f_1(z), \ldots, f_r(z)$ is an algebraic relation over $\overline{\mathbb{Q}}(z)$ between these functions and their successive derivatives.

A σ_q-algebraic relation between $f_1(z), \ldots, f_r(z)$ is an algebraic relation over $\overline{\mathbb{Q}}(z)$ between these functions and their successive images by $\sigma_q : z \to z^q$.

General Lifting Theorem

Let $f_1(z), \ldots, f_r(z)$ be $\star$-functions. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ be such that these functions are well-defined. Then any homogeneous algebraic relation over $\overline{\mathbb{Q}}$ between $f_1(\alpha), \ldots, f_r(\alpha)$ is the specialization of a homogeneous $\star$-algebraic relation over $\overline{\mathbb{Q}}(z)$ between $f_1(z), \ldots, f_r(z)$.

- For each $f_i(z)$, consider an E-system associated with an E-operator.
  Take the direct sum of these systems and apply the lifting theorem.

- For each $f_i(z)$, one can construct an M-system regular at α and involving only $f_i(z), f_i(z^q), \ldots$
  Take the direct sum of these systems and apply the lifting theorem.

Consequence

Descent Theorem

Let $f_1(z), \dots, f_r(z)$ be $\star$-functions with coefficients in a number field K. Let $\alpha \in K \setminus \{0\}$ be a point where these functions are well-defined. Then

$$\mathrm{Lin}_{\overline{\mathbb{Q}}}(f_1(\alpha), \dots, f_r(\alpha)) = \mathrm{Span}_{\overline{\mathbb{Q}}}(\mathrm{Lin}_K(f_1(\alpha), \dots, f_r(\alpha)))\,.$$

Corollary

Let $f(z)$ be a $\star$-function with coefficients in a number field K. Let $\alpha \in K \setminus \{0\}$ be a point where this functions is well-defined. Then either $f(\alpha)$ is transcendental of $f(\alpha) \in K$.

In the case of M-functions and $K = \mathbb{Q}$, this result has an important consequence concerning the complexity of the integer base expansions of algebraic numbers (Problem **P1**).

Divergence: algebraic relations at different points

To study the algebraic relation between $f_1(\alpha_1), \ldots, f_r(\alpha_r)$, it is sufficient to consider the algebraic relations between the values of the E-functions $f_1(\alpha_1 z), \ldots, f_r(\alpha_r z)$ at $z = 1$.

This more general problem rests on the previous theory!

Let $\mathcal{E}_\alpha$ be the smallest field generated by all the values of E-functions evaluated at α.

If α and β are two non-zero algebraic numbers, then

$$\mathcal{E}_\alpha = \mathcal{E}_\beta =: \mathcal{E} \,.$$

To study the algebraic relations between $f_1(\alpha_1), \ldots, f_r(\alpha_r)$, one has to develop Mahler's method in several variables.

Indeed, if $f(z)$ is an M-function and $\alpha \in \overline{\mathbb{Q}}$, then $f(\alpha z)$ is usually not an M-function!

Let $\mathcal{M}_\alpha$ be the smallest field generated by all the values of M-functions evaluated at α.

Theorem

If α and β are multiplicatively independent algebraic numbers, then $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are free over $\overline{\mathbb{Q}}$.

Conjecture

For all $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the fields $\mathcal{E}$ and $\mathcal{M}_\alpha$ are free over $\overline{\mathbb{Q}}$.