Motivic Galois theory for algebraic Mellin transforms

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1. Algebraic Mellin transforms

2. Twisted cohomology

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Algebraic Mellin transforms
Algebraic Mellin transforms

(Not in this talk) The classical Mellin transform (Mellin, 1897)

\[ \varphi : (0, \infty) \to \mathbb{C} \quad \mapsto \quad (\mathcal{M}\varphi)(s) = \int_0^\infty x^s \varphi(x) \frac{dx}{x}. \]

Algebraic Mellin transforms (Aomoto, 1974)

\[ I(s) = \int_\sigma f^s \omega. \]

- \( X \) an (affine, smooth) algebraic variety over a field \( k \subset \mathbb{C} \).
- \( f : X \to \mathbb{G}_m \) an invertible function on \( X \).
- \( \omega \) an algebraic differential form on \( X \), \( \sigma \) a topological cycle on \( X \).

(Bloch–Vlasenko call them “motivic Mellin transforms” or “motivic \( \Gamma \)-functions”.)

More generally, for \( f = (f_1, \ldots, f_N) : X \to \mathbb{G}_m^N \), consider multivariate versions:

\[ I(s_1, \ldots, s_N) = \int_\sigma f_1^{s_1} \cdots f_N^{s_N} \omega. \]
Examples of algebraic Mellin transforms

Example: the beta function

\[ B(s, t) = \int_0^1 x^s (1-x)^t \frac{dx}{x(1-x)} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \]

Corresponds to \((x, 1-x) : \mathbb{P}^1 \setminus \{\infty, 0, 1\} \longrightarrow \mathbb{G}_m^2.\)

Example: the classical hypergeometric function

\[ B(b, c-b) \, _2F_1(a, b; c; z) = \int_0^1 x^b (1-x)^{c-b}(1-zx)^{-a} \frac{dx}{x(1-x)} \]

Corresponds to \((x, 1-x, 1-zx) : \mathbb{P}^1 \setminus \{\infty, 0, 1, z^{-1}\} \longrightarrow \mathbb{G}_m^3.\)

Example: Feynman integrals

\( \Gamma \) a connected graph with \(n\) edges and first Betti number \(h\).

\[ I_\Gamma(\varepsilon) = \int_{\sigma_\Gamma} \left( \frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma^h} \right)^\varepsilon \omega_\Gamma \]

Corresponds to \(\mathbb{P}^{n-1} \setminus \{\Psi_\Gamma \Xi_\Gamma = 0\} \longrightarrow \mathbb{G}_m.\)
Arithmetic structure of algebraic Mellin transforms

(Not in this talk) Systems of finite difference equations

\[ l_i(s + 1) = \sum_{i=1}^{n} f_{i,j}(s) l_j(s) \quad \text{with} \quad f_{i,j}(s) \in k(s). \]

▶ Example: \( B(s + 1, t) = \frac{s}{s + t} B(s, t) \), \( B(s, t + 1) = \frac{t}{s + t} B(s, t) \).

▶ Corresponds to a rank 1 “finite difference module” (Loeser–Sabbah).

(Not in this talk) Values at \( s \in \mathbb{Q} \)

For \( s \in \mathbb{Q} \), \( I(s) \) is a period of a cyclic cover of \( X \).

(In this talk) Laurent expansion at \( s = 0 \)

\[ I(s) = \sum_{n \gg -\infty} \alpha_n s^n \quad \text{where the} \quad \alpha_n \quad \text{are periods.} \]

We are interested in the Galois theory of the \( \alpha_n \).
The slogan
Galois theory of algebraic numbers should extend to a Galois theory for periods, where the Galois groups are algebraic groups over \( \mathbb{Q} \).

- Periods arise as coefficients of the perfect pairing
  \[
  \int : H^n_B(X) \times H^n_{dR}(X) \longrightarrow \mathbb{C} , \quad (\sigma, \omega) \mapsto \int_\sigma \omega
  \]
  for algebraic varieties \( X \), or pairs \((X, Y)\), defined over \( \mathbb{Q} \).

- A tannakian formalism of motives gives rise to a motivic Galois group that acts linearly on all \( H^n_{dR}(X) \) and \( H^n_{dR}(X, Y) \).

- This gives rise to a Galois theory for periods:
  
  \[
  "g \cdot \int_\sigma \omega := \int_\sigma g.\omega"
  \]

- Grothendieck’s period conjecture says that this formula is well-defined.

- Unconditional: Galois theory for motivic periods.

- Computable: Galois coaction.
The key example: the beta function

- Not great: \[ B(s, t) = \frac{s + t}{st} \left(1 - \sum_{m,n \geq 1} (-s)^m (-t)^n \zeta(1, \ldots, 1, m + 1) \right). \]

- Better:

\[
B(s, t) = \frac{s + t}{st} \exp \left( \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) (s^n + t^n - (s + t)^n) \right).
\]

- Galois theory for zeta values: for \( g \) in the motivic Galois group,

\[ g \cdot \zeta(n) = \zeta(n) + a_g^{(n)} \quad \text{with} \quad a_g^{(n)} \in \mathbb{Q}. \]

- We get a Galois theory for the beta function:

\[ g \cdot B(s, t) = A_g(s, t) B(s, t) \quad \text{with} \quad A_g(s, t) \in \mathbb{Q}((s, t))^\times. \]

- \( B(s, t) \) corresponds to a rank 1 representation of the motivic Galois group defined over \( \mathbb{Q}((s, t)) \).
The motivic Galois group acts on Taylor expansions of algebraic Mellin transforms via power series, i.e., for $g$ in the motivic Galois group $G$:

$$g \cdot \int_{\sigma} f^s \omega = \sum_{i=1}^{N} A_{g}^{(i)}(s) \int_{\sigma} f^s \omega_i$$

where the $A_{g}^{(i)}(s)$ are in $k((s))$.

▶ This is a finite formula which computes the Galois theory of infinitely many periods.
Proof of concept

► A rank 2 example:

\[ I(a; s) = \frac{1}{s} \left( _2F_1(s, 1, s + 1; a) - 1 \right) = \int_0^1 x^s \frac{a \, dx}{1 - ax} = \sum_{n=0}^{\infty} (-s)^n \text{Li}_{n+1}(a). \]

Galois theory:

\[ g.l(a; s) = A_g(a; s) I(a; s) + B_g(a; s) \quad \text{with} \quad A_g(a; s), B_g(a; s) \in \mathbb{Q}((s)). \]

► A family of examples (Brown-D. 2022): Lauricella hypergeometric functions

\[ \int_0^{\sigma_j} x^{s_0} (1 - x\sigma^{-1}_{s_1})^{s_1} \cdots (1 - x\sigma^{-1}_{s_n})^{s_n} \frac{dx}{x - \sigma_j}. \]
Twisted cohomology
Twisted cohomology

$X$ an (affine, smooth) algebraic variety over $\mathbb{C}$, $f : X \to \mathbb{C}^*$.

$$H^\bullet(X, f) := H^\bullet(X, f^*(t^s)).$$

- Fix $s \in \mathbb{C}$.
- de Rham: $H^i_{dR}(X, f) := H^i(X, (\Omega^\bullet_X, \nabla_s))$ where
  $$\nabla_s : \omega \mapsto d\omega + s \frac{df}{f} \wedge \omega \quad \text{(so that $d(f^s \omega) = f^s \nabla_s(\omega)$)}.$$
- Betti: $H^i_B(X, f) := H^i_{sing}(X, L_s)$ where
  $$L_s = \mathbb{C} f^s \quad \text{(monodromy $e^{2\pi i s}$)}.$$
- Algebraic Mellin transforms arise as coefficients of the perfect pairing
  $$\int : H^i_B(X, f) \times H^i_{dR}(X, f) \to \mathbb{C}, \quad (\sigma, \omega) \mapsto \int_\sigma f^s \omega.$$
- Easy to compute for generic values of $s \in \mathbb{C}$. Typical behavior:
  $$\begin{cases}
  H^i(X, f) = 0 & \text{for } i \neq n := \dim(X); \\
  \dim H^n(X, f) = (-1)^n \chi(X).
  \end{cases}$$
Does twisted cohomology come from geometry?

- $H^\bullet(X, f)$ is not motivic (does not come from geometry) if $s \not\in \mathbb{Q}$.
- A formal generic version of $H^\bullet(X, f)$ is motivic (comes from geometry).

- de Rham: a finite dimensional vector space over $k((s))$,
  \[ M_{dR}^i(X, f) := H^i(X, (\Omega_X^\bullet((s)), \nabla)), \]
  where $\nabla : \omega \mapsto d\omega + s \frac{df}{f} \wedge \omega$.

- Betti: a finite dimensional vector space over $\mathbb{Q}((\log \mu))$,
  \[ M^B_i(X, f) := H^i_{\text{sing}}(X, \mathcal{L}), \]
  where $\mathcal{L}$ is the rank 1 local system of vector spaces over $\mathbb{Q}((\log \mu))$
  \[ \pi_1(X(\mathbb{C})) \xrightarrow{f_*} \pi_1(\mathbb{C}^*) = \mathbb{Z} \xrightarrow{\mu} \mathbb{Q}((\log \mu))^\times \]

- Perfect pairing valued in $\mathbb{C}((s))$, with $\mu \leftrightarrow e^{2\pi is}$, giving rise to Laurent expansions of algebraic Mellin transforms.
Why is twisted cohomology motivic?

\[ \text{M}^i_{\text{dR}}(X, f) := H^i(X, (\Omega^\bullet_X((s)), \nabla)) \]

\[ \simeq \left( \lim_{\to} H^i(X, (\Omega^\bullet_X[s]/(s^{n+1}), \nabla)) \right) \otimes_{k[[s]]} k((s)). \]

\[ =: M_{n, \text{dR}} \]

- Analogy with étale \( \ell \)-adic cohomology:

\[ H^\bullet_{\text{ét}}(X; \mathbb{Q}_\ell) := \left( \lim_{\to} H^\bullet_{\text{ét}}(X; \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \]

Each \( M_{n, \text{dR}} \) is motivic

- Comes from the motivic fundamental group of \( \mathbb{G}_m \) (Hain, Deligne).
- The \( k[s]/(s^{n+1}) \)-module structure is motivic, where \( s \leftrightarrow H_1(\mathbb{G}_m) \).
- Tannakian category of “local Mellin motives”

\[ M(X, f) = (\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0). \]
Application to Feynman integrals
Feynman integrals

- $\Gamma$ a connected graph with $n$ edges and first Betti number $h$.
- Graph polynomials $\Psi_\Gamma$, $\Xi_\Gamma$, homogeneous in $n$ variables.
- Feynman integral

$$I_\Gamma = \int_{\mathbb{P}^{n-1}(\mathbb{R}_+)} \frac{\Psi_{\Gamma}^{n-(h+1)D/2}}{\Xi_{\Gamma}^{n-hD/2}} \Omega.$$  

- $\Xi_\Gamma/\Psi_\Gamma$ is a “tropical height” (Amini–Bloch–Burgos–Fresán, Tourkine).

Example: the massless triangle graph ($D = 4$)

$$I_\Gamma = \int\int_{(0,\infty)^2} \frac{dx \, dy}{(x + y + 1)(q_1^2x + q_2^2y + q_3^2xy)}$$

![Massless Triangle Graph](image-url)
Problem: Feynman integrals *do not always converge!*

A wild idea

Work in space-time dimension

\[ D = 4 - 2\varepsilon \]

and consider the Laurent expansion near \( \varepsilon = 0 \).

Example: the massless triangle graph

\[
I_\Gamma(\varepsilon) = \int_{(0,\infty)^2} \left( \frac{(x + y + 1)^2}{q_1^2x + q_2^2y + q_3^2xy} \right)^\varepsilon \frac{dx dy}{(x + y + 1)(q_1^2x + q_2^2y + q_3^2xy)}
\]

▶ This is an algebraic Mellin transform for

\[ f = \frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma} : X = \mathbb{P}^{n-1} \setminus \{ \Psi_\Gamma \Xi_\Gamma = 0 \} \rightarrow \mathbb{G}_m. \]

▶ Corresponding geometry: \((X, \bigcup_i \{ x_i = 0 \}, f)\).
Theorem (Brown–D.–Fresán–Tapušković)

The space of Laurent expansions of Feynman integrals in dimensional regularization is closed under the action of the motivic Galois group:

\[ g \cdot I_{\Gamma}(\varepsilon) = \sum_{i=1}^{N} A_g^{(i)}(\varepsilon) I_{\Gamma_i}(\varepsilon) \quad \text{with} \quad A_g^{(i)}(\varepsilon) \in \overline{\mathbb{Q}}((\varepsilon)). \]

- Still difficult to make explicit.
- Should lead to arithmetic constraints on Feynman integrals.