A Lindemann–Weierstrass theorem for E-functions

Atelier E-fonctions, G-fonctions et périodes

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Structure of the talk

Structure of the talk

Diophantine properties of the exponential function

Hermite–Lindemann (1873–1882)

If $\alpha \in \overline{\mathbb{Q}}$ is non-zero, then e^{α} is transcendental.

Question 1

Let f be an E-function and $\alpha \in \overline{\mathbb{Q}}$ non-zero. Is $f(\alpha)$ transcendental ?

Completely effectively solved by Adamczewski–Rivoal (2018) with refinements by Bostan–Rivoal–Salvy (2022). Based on

- André's theory of E-operators (2000)
- Beukers' refinement of Siegel-Shidlovskii theorem (2006)

Diophantine properties of the exponential function

Lindemann–Weierstrass (1882–1885)

If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are distinct, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-linearly independent.

Question 2

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ distinct. Are $f(\alpha_1), \dots, f(\alpha_n)$ $\overline{\mathbb{Q}}$-linearly independent ?

Lindemann–Weierstrass

If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

Question 3

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ be $\mathbb{Q}$-linearly independent. Are $f(\alpha_1), \dots, f(\alpha_n)$ algebraically independent over $\mathbb{Q}$?

Foundational results

André (2000) : E-operators

Every E-function is annihilated by a differential operator which has only 0 and ∞ as possible singularities.

Beukers (2006) : Refinement of the Siegel–Shidlovskii theorem

Let $Y = (f_1, \dots, f_n)^\top$ be a vector of E-functions satisfying $Y' = AY$ where A is an $n \times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote the common denominator of the entries of A by $T(z)$.

Let $\alpha \in \overline{\mathbb{Q}}$ satisfying $\alpha T(\alpha) \neq 0$.

Then, for any homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ such that $P(f_1(\alpha), \dots, f_n(\alpha)) = 0$, there exists a polynomial $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the variables $X_1, \dots, X_n$, such that $Q(\alpha, X_1, \dots, X_n) = P(X_1, \dots, X_n)$ and $Q(z, f_1(z), \dots, f_n(z)) = 0$.

Structure of the talk

Question 1

Let f be an E-function and $\alpha \in \overline{\mathbb{Q}}$ non-zero. Is $f(\alpha)$ transcendental ?

Exceptional set

$$\mathrm{Exc}(f) = \{\alpha \in \overline{\mathbb{Q}}^\times \ : \ f(\alpha) \in \overline{\mathbb{Q}}\}.$$

We say that f is purely transcendental if $\mathrm{Exc}(f)$ is empty.

- Hermite–Lindemann : exp is purely transcendental.
- Siegel (1929) : Bessel function J_0 is purely transcendental :

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n}.$$

- Bostan–Rivoal–Salvy (2022) : $|\mathrm{Exc}(f)| = d$

$$f(x) = \sum_{n=0}^{\infty} \binom{n+d}{d} \frac{1}{(a+d+1)_n} z^n, \quad (a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}).$$

Canonical decomposition

- An algebraic E-function is polynomial.
- If f is purely transcendental and $p, q \in \overline{\mathbb{Q}}[z]$, then $\mathrm{Exc}(p + qf)$ is the set of the non-zero roots of q.
- This is in fact the only way to construct non-purely transcendental E-functions !

Rivoal (2016), Bostan-Rivoal-Salvy (2022)

Every transcendental E-function f can be written in a unique way as $f = p + qg$ with $p, q \in \overline{\mathbb{Q}}[z]$, q monic, $q(0) \neq 0$, $\deg(p) < \deg(q)$ and g is a purely transcendental E-function.

***HL* algorithm by Adamczewski–Rivoal (2018)**

It takes an E-function f as input. It first says whether f is transcendental or not. Then it returns $\mathrm{Exc}(f)$ if f is transcendental.

Remark : improved by Bostan–Rivoal–Salvy (2022). This yields an algorithm to compute canonical decompositions.

Structure of the talk

Question 2

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ distinct. Are $f(\alpha_1), \dots, f(\alpha_n)$ $\overline{\mathbb{Q}}$-linearly independent ?

- First obstruction : $\alpha_1, \dots, \alpha_n$ must be non-exceptional values for f.
- Second obstruction : difference equations such as $J_0(-x) = J_0(x)$.

Siegel (1929)

If $\alpha_1^2, \dots, \alpha_n^2$ are pairwise distinct non-zero algebraic numbers, then $J_0(\alpha_1), \dots, J_0(\alpha_n)$ are algebraically independent over $\mathbb{Q}$.

On Beukers' lifting result

Let $Y = (f_1, \ldots, f_n)^\top$ be a vector of $\overline{\mathbb{Q}}(z)$-linearly independent E-functions.

$$Y' = AY.$$

Write $T(z)$ for the common denominator of the entries of A. Let $\alpha \in \overline{\mathbb{Q}}$.

When $\alpha T(\alpha) \neq 0$

Then $f_1(\alpha), \ldots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linearly independent.

When $T(\alpha) = 0$

There exists a non-trivial $\overline{\mathbb{Q}}$-linear relation between $f_1(\alpha), \ldots, f_n(\alpha)$.

Indeed, write $B(z) = T(z)A(z)$. Then $T(z)Y'(z) = B(z)Y(z)$ yields $0 = B(\alpha)Y(\alpha)$.
Even more : Every such relation can be explicitely given by the system through a desingularization process by Beukers.

Lindemann–Weierstrass and Beukers' lifting result

Lindemann–Weierstrass

Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers and consider $f_i(z) = e^{\alpha_i z}$.

$$\begin{pmatrix} e^{\alpha_1 z} \\ \vdots \\ e^{\alpha_n z} \end{pmatrix}' = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} e^{\alpha_1 z} \\ \vdots \\ e^{\alpha_n z} \end{pmatrix}.$$

We have $T(z) = 1$ so $T(1) \neq 0$ and $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-linearly independent.

Problem : Given an E-function f and $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$:

- Are $f(\alpha_1 z), \ldots, f(\alpha_n z)$ $\overline{\mathbb{Q}}(z)$-linearly independent ?
- Is $T(1)$ non-zero ?

How to choose n and $\alpha_1, \ldots, \alpha_n$ to answer yes twice ?

Singularities of the underlying G-function

To every E-function corresponds a G-series $\psi(f)$ defined by

$$\psi\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n\right)=\sum_{n=0}^{\infty}a_n z^n.$$

In particular, $\psi(f)$

- satisfies a linear differential equation over $\overline{\mathbb{Q}}[z]$,
- has a positive radius of convergence,
- has finitely many singularities at finite distance, the set of which we denote by $\mathfrak{S}(f)$.

Examples

- $\psi(\exp)=1/(1-z)$ and $\mathfrak{S}(\exp)=\{1\}$.
- $\psi(J_0)=1/\sqrt{1+z^2}$ and $\mathfrak{S}(J_0)=\{-i,i\}$.

A Lindemann–Weierstrass theorem for E-functions

D. (2022)

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ non-zero such that

- for all i, $\alpha_i \notin \mathrm{Exc}(f)$,
- for all $i \neq j$ and all $\rho_1, \rho_2 \in \mathfrak{S}(f)$, $\alpha_i/\alpha_j \neq \rho_1/\rho_2$.

Then $1, f(\alpha_1), \dots, f(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.

Exponential function

exp is purely transcendental and $\mathfrak{S}(\exp) = \{1\}$. The second condition reads $\alpha_i \neq \alpha_j$: we retrieve the Lindemann–Weierstrass theorem.

Bessel function

J_0 is purely transcendental and $\mathfrak{S}(J_0) = \{-i, i\}$. The second condition reads $\alpha_i^2 \neq \alpha_j^2$. We retrieve the linear part of Siegel's result.

Structure of the talk

Entire hypergeometric functions

We consider (entire) hypergeometric functions

$$F(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} z^n,$$

where $s > r \geq 0$, $a_i, b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $(a)_n$ denotes the Pochhammer symbol defined by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$.

D. (2022)

Let F be a non-polynomial hypergeometric function with $s > r$ and rational parameters. Let $\alpha_1, \ldots, \alpha_n$ be pairwise non-zero distinct algebraic numbers which are not exceptional values for F. Then $1, F(\alpha_1), \ldots, F(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.

Hints : Write $k = s - r$ and consider the E-function $f(z) = F(z^k)$.

- $\psi(f) = H(kz^k)$ where H is a hypergeometric G-function.
- Elements of $\mathfrak{S}(f)$ are of the form ρ/k where ρ is a k-th root of unity.
- if $\alpha_i = \beta_i^k$, then $\alpha_i \neq \alpha_j$ implies $\beta_i/\beta_j \neq \rho_1/\rho_2$.

A non-hypergeometric E-function by Fresán and Jossen

Consider

$$f(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor 2n/3 \rfloor} \frac{(1/4)_{n-m}}{(2n-3m)!(2m)!} z^n.$$

The following calculations were done by Alin Bostan :

- A linear differential equation satisfied by f.
- Application of the Bostan–Rivoal–Salvy implementation of Adamczewski–Rivoal algorithm : f is purely transcendental.
- A differential operator annihilating $\psi(f)$ with a minimal number of singularities : the roots ρ_1, ρ_2 and ρ_3 of

$$23z^3 + 128z^2 + 128z - 256.$$

- If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are non-zero and such that $\alpha_i/\alpha_j \neq \rho_k/\rho_\ell$, then $1, f(\alpha_1), \dots, f(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.

Structure of the talk

General statement

D. (2022)

Let $f_1, \dots, f_n$ be E-functions with pairwise disjoint sets $\mathfrak{S}(f_i)$. Let $\alpha \in \overline{\mathbb{Q}}$ non-zero be such that $\alpha \notin \mathrm{Exc}(f_i)$ for all i. Then $1, f_1(\alpha), \dots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linearly independent.

Corollary : $f_i(x) = f(\alpha_i x)$ with $\alpha_i/\alpha_j \neq \rho_1/\rho_2$ and $\alpha = 1$.

Hints : By contradiction. Set $f_0 = 1$ and :

- Consider a non-trivial relation $\lambda_0 f_0(\alpha) + \cdots + \lambda_n f_n(\alpha) = 0$.
- André's theory of E-operators : $f_0, \dots, f_n$ together with some derivatives form a vector solution of a system $Y' = AY$ with only 0 and ∞ as singularities : $\alpha T(\alpha) \neq 0$.
- Beuker's lifting result : there are linear differential operators $\mathcal{L}_i$ with coeff. in $\overline{\mathbb{Q}}[z]$ s.t. $\mathcal{L}_0 f_0 + \cdots + \mathcal{L}_n f_n = 0$ and $(\mathcal{L}_i f_i)(\alpha) = \lambda_i f_i(\alpha)$.
- Laplace transform : $\psi(\mathcal{L}_0 f_0) + \cdots + \psi(\mathcal{L}_n f_n) = 0$.

$$\psi(\mathcal{L}_0 f_0) + \cdots + \psi(\mathcal{L}_n f_n) = 0 \quad \text{and} \quad (\mathcal{L}_i f_i)(\alpha) = \lambda_i f_i(\alpha).$$

Lemma

The singularities of $\psi(\mathcal{L}_i f_i)$ are singularities of $\psi(f_i)$.

Hint : By induction on the order and the degree of $\mathcal{L}_i$:

$$\psi(zf(z)) = \left(z^2 \frac{d}{dz} + z\right)\psi(f) \quad \text{and} \quad \psi\left(\frac{d}{dz} f(z)\right) = \frac{\psi(f)(z) - f(0)}{z}.$$

- The $\psi(\mathcal{L}_i f_i)$ have distinct singularities at finite distance.
- So $\psi(\mathcal{L}_i f_i)$ has no singularity at finite distance !
- G-functions with no singularity at finite distance are polynomial.
- Hence $\mathcal{L}_i f_i \in \overline{\mathbb{Q}}[z]$ for all i.
- for all i, $\lambda_i f_i(\alpha) = (\mathcal{L}_i f_i)(\alpha) \in \overline{\mathbb{Q}}$.
- $\lambda_i \neq 0$ for at least one i : $f_i(\alpha) \in \overline{\mathbb{Q}}$, a contradiction.

□

What about the second formulation ?

Question 3

Let f be an E-function and $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ linearly independent over $\mathbb{Q}$.
Are $f(\alpha_1), \ldots, f(\alpha_n)$ algebraically independent over $\mathbb{Q}$?

In the case of $f = \exp$, the Siegel–Shidlovskii theorem is sufficient because $\exp(\alpha_1 z), \ldots, \exp(\alpha_n z)$ are algebraically independent over $\overline{\mathbb{Q}}[z]$.

Thank you for your attention !