Introduction to E-functions

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Atélier E-fonctions, G-fonction et périodes
Paris, 9 Janvier 2023

A general transcendence problem

Let $f(z) \in \overline{\mathbb{Q}}[[z]]$ be power series in z with coefficients in $\overline{\mathbb{Q}}$, with positive radius of convergence ρ. We assume $f(z)$ is not algebraic over $\overline{\mathbb{Q}}(z)$.

Question

Let $\alpha \in \overline{\mathbb{Q}}$ and suppose $0 < |\alpha| < \rho$. Is $f(\alpha)$ transcendental?

A bizarre function

There exist non-algebraic $f \in \mathbb{Q}[[z]]$ with $\rho = \infty$ such that

$$f(\alpha) \in \overline{\mathbb{Q}} \text{ for all } \alpha \in \overline{\mathbb{Q}}$$

Idea of construction:
Enumerate the elements of $\mathbb{Z}[z]$ by $P_1, P_2, \dots$ and consider

$$f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k P_1(z) \cdots P_k(z)$$

where $c_k \in \mathbb{Q}$ are chosen such that the resulting f has infinite radius of convergence.
Most of the following (and much more!) can be found in Tanguy Rivoal's survol

https://rivoal.perso.math.cnrs.fr/articles/EGxups.pdf

Lindemann-Weierstrass theorem

Around 1882 F.Lindemann proved the transcendence of π. In fact his method yielded more.

Theorem (Lindemann-Weierstrass)

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct algebraic numbers. Then

$$e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$$

are linearly independent over $\overline{\mathbb{Q}}$.

Application: π is transcendental.
Proof: Suppose π were algebraic. Take $\alpha_1 = 0, \alpha_2 = \pi i$. Then Lindemann-Weierstrass implies that $1, e^{\pi i}$ are $\overline{\mathbb{Q}}$-linear independent, contradicting $e^{\pi i} = -1$.

E-function definition

Definition

An entire function $f(z)$ given by a powerseries

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

with $a_k \in \overline{\mathbb{Q}}$ for all k, is called an E-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.
2. The height $H(a_0, a_1, \ldots, a_k)$ is bounded by an exponential bound of the form C^k, where $C > 0$ depends only on f.

Remark: Siegel formulated $H(a_1, \ldots, a_k) = O_\epsilon(k!^\epsilon)$ for all $\epsilon > 0$ in his definition. We speak of E-functions in the *broad sense* in that case.

E-function examples

$$\begin{aligned}
\exp(az) &= \sum_{k=0}^{\infty} \frac{a^k z^k}{k!},\ a \in \overline{\mathbb{Q}}^{\times} \\
J_0(-z^2) &= \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!} = \sum_{k\geq 0} \binom{2k}{k} \frac{z^{2k}}{(2k)!} \\
P(z) &\in \overline{\mathbb{Q}}[z] \quad \text{(trivial case)}
\end{aligned}$$

The corresponding differential equations read

$$\begin{aligned}
y' - ay &= 0 \\
zy'' + y' - 4zy &= 0 \\
P(z)y' - P'(z)y &= 0
\end{aligned}$$

Hypergeometric example

A very general example, the *confluent hypergeometric series*,

$$ {}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\,\middle|\, z^{q+1-p}\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k k!} z^{(q+1-p)k} $$

where $q \geq p$ (confluence) and $\alpha_i, \beta_j \in \mathbb{Q}$ for all i, j.
$(x)_n$ is the Pochhammer symbol defined by $x(x+1)\cdots(x+n-1)$.
${}_pF_q$ satisfies a linear differential equation of order $q+1$.

Differential ring structure

The E-functions form a so-called differential ring. More precisely,

Proposition

Let $f(z), g(z)$ be E-functions. Then the following functions are again E-functions

- $f'(z)$
- $f(z) + g(z)$
- $f(z)g(z)$

Theorem (Y.André)

The units in the ring of E-functions are given by $\beta e^{\alpha z}$ with $\alpha, \beta \in \overline{\mathbb{Q}}$, $\beta \neq 0$.

First order systems

Let L be any algebraic number field. An $n \times n$-system of first order linear differential equations over L is given by

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

where $A_{ij} \in L(z)$ for all i, j.
We abbreviate by

$$\mathbf{y}' = A\mathbf{y}$$

where A is the $n \times n$-matrix with entries A_{ij}.
Let $T(z)$be the common denominator of the A_{ij}. The zeros of $T(z)$ are called the *singularities* of the system.

From equations to systems

Consider the linear n-th order differential equation

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y' + p_n y = 0, \;\; p_i \in L(z)$$

Put

$$y_1 = y, \;\; y_2 = y', \;\; \ldots, \;\; y_n = y^{(n-1)}$$

Note that

$$y_1' = y_2, \;\; y_2' = y_3, \;\; \ldots, \;\; y_{n-1}' = y_n.$$

Finally,

$$y_n' = -p_1 y_n - p_2 y_{n-1} - \ldots - p_n y_1.$$

Rewrite as

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Siegel-Shidlovskii theorem

Siegel-Shidlovskii, 1929, 1956

Let $(f_1(z), \dots, f_n(z))^t$ be a solution vector of a system of first order equations of the form

$$\mathbf{y}'(z) = A(z)\mathbf{y}(z)$$

and suppose that the $f_i(z)$ are E-functions. Let $T(z)$ be the common denominator of the entries of $A(z)$. Let $\alpha \in \overline{\mathbb{Q}}$ and suppose $\alpha T(\alpha) \neq 0$. Then

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha)) = \mathrm{degtr}_{\mathbb{C}(z)}(f_1(z), f_2(z), \dots, f_n(z))$$

In particular, if the $f_i(z)$ are algebraically independent over $\mathbb{C}(z)$ then the values at $z = \alpha$ are algebraically independent over $\overline{\mathbb{Q}}$ (or $\mathbb{Q}$, which amounts to the same).

Algebraic relations between E-function

In the 1960's and 70's much energy has gone into showing algebraic independence of (mainly hypergeometric) E-functions.
In the 1980's the tool of differential galois theory was used, which clarified very much of the earlier work.
Example of a relation:
Let $r \in \mathbb{Z}_{>1}$ be odd and consider

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^r} z^k.$$

Then $f(z^r)$ is an E-function satisfying a differential equation of order r.
A bilinear relation,

$$\sum_{i=0}^{r-1} (-1)^i \times \theta^i f(z) \times \theta^{r-1-i} f(-z) = 0,$$

where $\theta = z\frac{d}{dz}$.

Another relation

Let $r \in \mathbb{Z}_{\geq 1}$ be odd. Define

$$f(z) = \sum_{k=0}^{\infty} \frac{(1/2)_k}{(k!)^r} z^k.$$

Then $f(z^{r-1})$ is an E-function satisfying a differential equation of order r. The differential galois group has the form $C.SO(r, \mathbb{C})$.
A quadratic relation,

$$\sum_{i=0}^{r-1} (-1)^i \times \theta^i f(z) \times \theta^{r-1-i} f(z) = z f(z)^2,$$

where $\theta = z\frac{d}{dz}$.

Exceptional Galois groups

Consider

$$f(z) = \sum_{k \geq 0} \frac{(1/14)_k}{(7k)!} z^k.$$

Solution of 7th order differential equation. Katz showed that its Galois group equals $G_2 \times \mathbb{Z}/7\mathbb{Z}$.
Some further special Galois groups for suitable parameters (Katz),

- $q = 8, p = 2, G = C.SL(3)$ (adjoint representation)
- $q = 8, p = 2, G = C.(SL(2) \times SL(2) \times SL(2))$
- $q = 8, p = 2, G = C.(SL(2) \times Sp(4))$
- $q = 8, p = 2, G = C.(SL(2) \times SL(4))$
- $q = 9, p = 3, G = C.(SL(3) \times SL(3))$

Refining Shidlovskii

Nesterenko-Shidlovskii,1996

Let $f_1, f_2, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations. Then there exists a finite set $S \subset \overline{\mathbb{Q}}$ such that for any algebraic number α not in S, polynomial relations over $\overline{\mathbb{Q}}$ between the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ arise from specialization of polynomial relations over $\overline{\mathbb{Q}}(z)$ of the same degree between the functions $f_1(z), \ldots, f_n(z)$.

Proof uses the Siegel-Shidlovskii method.

Theorem (FB (2006), Y.André (2014))

For the exceptional set S one can take the zero set of $zT(z)$.

Remark: André's 2014 proof also holds for E-functions in the broad sense and can be extended to discrete analogues of Siegel-Shidlovskii.

Application to linear (in)dependence

Corollary

Let $f_1, f_2, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations with singularities given by $T(z) = 0$. Let $\alpha \in \overline{\mathbb{Q}}$ with $\alpha T(\alpha) \neq 0$. Then any linear relation over $\overline{\mathbb{Q}}$ between the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ arises from specialization of a linear relation over $\overline{\mathbb{Q}}(z)$ between the functions $f_1(z), \ldots, f_n(z)$ (former question of Lang).

Corollary

In particular, $\overline{\mathbb{Q}}(z)$-linear indepence of the $f_i(z)$ implies $\overline{\mathbb{Q}}$-linear independence of the $f_i(\alpha)$ when $\alpha T(\alpha) \neq 0$.

Reduction of E-functions

Theorem (FB 2006)

Let $f_1, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations with singularities given by $T(z) = 0$. Suppose they are $\overline{\mathbb{Q}}(z)$-linear independent. Then there exist E-functions $e_1(z), \ldots, e_n(z)$ and an $n \times n$-matrix $M(z)$ with entries in $\overline{\mathbb{Q}}[z]$ such that

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = M \begin{pmatrix} e_1(z) \\ \vdots \\ e_n(z) \end{pmatrix}$$

and where $(e_1(z), \ldots, e_n(z))$ is vector solution of a system of n homogeneous first order equations with coefficients in $\overline{\mathbb{Q}}[z, 1/z]$.

Dependence relations are 'trivial'

Corollary

Suppose $\alpha \in \overline{\mathbb{Q}}^\times$ and $b_1, \ldots, b_n \in \overline{\mathbb{Q}}$. Then

$$b_1 f_1(\alpha) + b_2 f_2(\alpha) + \cdots + b_n f_n(\alpha) = 0$$

if and only if $(b_1, b_2, \ldots, b_n) M(\alpha) = 0$.

Proof: Suppose $b_1 f_1(\alpha) + \cdots + b_n f_n(\alpha) = 0$. Then it follows from the theorem that

$$0 = (b_1, \ldots, b_n) M(\alpha) \begin{pmatrix} e_1(\alpha) \\ \vdots \\ e_n(\alpha) \end{pmatrix}.$$

Since $e_1(\alpha), \ldots, e_n(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent our corollary follows.

Remarks on linear differential equations

Consider a linear differential equation

$$q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0$$

where $q_i(z) \in \mathbb{C}[z]$ for all i.
The zeros of $q_n(z)$ are called the *singularities* of the equation, all other points are called *non-singular*.

Theorem, Cauchy

Suppose $a \in \mathbb{C}$ is a non-singular point. Then the solutions of the equation in $\mathbb{C}[[z-a]]$ form an n-dimensional $\mathbb{C}$-vector space. Furthermore there is an isomorphism of this space with $\mathbb{C}^n$ given by

$$y(z) \mapsto (y(a), y'(a), y''(a), \ldots, y^{(n-1)}(a)).$$

Finally, the solutions in $\mathbb{C}[[z-a]]$ all have positive radius of convergence.

Apparent singular points

- It may happen that there exists a basis of solutions in $\mathbb{C}[[z-a]]$ but a is a singularity. In that case we call a an *apparent singularity*.
- For example, if all solutions around $z=a$ have a zero a then a is an apparent singularity.
  In particular $y \mapsto (y(a), y'(a), \ldots, y^{(n-1)}(a))$ is not bijective any more.

Some more remarks

We abbreviate our equation

$$q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0$$

with $q_i(z) \in \mathbb{C}(z)$ by

$$Ly = 0$$

where $L \in \mathbb{C}(z)[d/dz]$ denotes the corresponding linear differential operator.

Let f be a function which satisfies a linear differential equation with coefficients in $\mathbb{C}(z)$. A *minimal differential equation* for f is an equation of smallest possible order satisfied by f.

Proposition

Let $Ly = 0$ be a mimimal differential equation for f. Then for any differential equation $L_1 y = 0$ satisfied by f there exists a differential operator L_2 such that $L_1 = L_2 \circ L$.

A miraculous theorem

Theorem, Y.André (2000)

Let $f(z)$ be an E-function. Then $f(z)$ satisfies a differential equation of the form

$$z^m y^{(m)} + \sum_{k=0}^{m-1} q_k(z) y^{(k)} = 0$$

where $q_k(z) \in \overline{\mathbb{Q}}[z]$ for all k.

- The equation from André's theorem need not be the minimal equation of $f(z)$.
- For example, the function $(z-1)e^z$ is an E-function, and its minimal differential equation reads $(z-1)f' = zf$. So we have a singularity at $z=1$. The equation refered to in André's theorem might be $f'' - 2f' + f = 0$.

A consequence

Corollary, Y.André 2000

Let f be an E-function with *rational* coefficients. Suppose that $f(1) = 0$. Then the minimal differential equation of f has an apparent singularity at $z = 1$.

The simplest example is again $f = (z-1)e^z$, an E-function which vanishes at $z = 1$. Its minimal differential equation is $(z-1)f' = zf$.

Proof of André's corollary

Proof. Consider $f(z)/(1-z)$. We will show that it is an E-function again. It is certainly an entire analytic function. Suppose that

$$f(z) = \sum_{r\geq 0} \frac{f_r}{r!} z^r, \quad f_r \in \mathbb{Q}$$

Then the power series of $f(z)/(1-z)$ reads

$$\frac{f(z)}{1-z} = \sum_{r\geq 0} \frac{g_r}{r!} z^r$$

where

$$g_r = r! \sum_{k=0}^{r} \frac{f_k}{k!}.$$

Suppose that the common denominator of $f_0, \ldots, f_r$ and the sizes $|f_r|$ are bounded by C^r for some $C > 0$. Then clearly the common denominators of $g_0, \ldots, g_r$ are again bounded by C^r.

End of proof

Recall

$$g_r = r! \sum_{k=0}^{r} \frac{f_k}{k!}.$$

To estimate the size of $|g_r|$ we use the fact that $0 = f(1) = \sum_{k \geq 0} f_k/k!$. More precisely,

$$\begin{aligned} |g_r| &= \left| -r! \sum_{k>r} f_k/k! \right| \\ &\leq \sum_{k>r} |f_k|/(k-r)! \\ &\leq \sum_{k>r} C^k/(k-r)! < C^r e^C \end{aligned}$$

So $|g_r|$ is exponentially bounded in r. Hence $f(z)/(1-z)$ is an E-function.

Final touch

- Notice that this argument only works if $f(z)$ is an E-function with *rational coefficients*, i.e. in $\mathbb{Q}$.
- By André's theorem $f(z)/(1-z)$ satisfies a differential equation without singularity at $z=1$.
- Hence its minimal differential equation has a full solution space V of analytic solutions at $z=1$.
- The solutions of the minimal equation of $f(z)$ can be found by multiplying the elements from V by $z-1$.
- This means that the minimal equation for $f(z)$ has a full space of analytic solutions all vanishing at $z=1$.
- So $z=1$ is apparent singularity.

Generalizing André's corollary

Using a combination of André's theorem and some differential galois theory one can prove the following result.

Theorem, FB (2006)

Let $f(z)$ be an E-function with coefficients in $\overline{\mathbb{Q}}$. Suppose that $f(1) = 0$. Then 1 is an apparent singularity of the minimal differential equation satisfied by f.

Another proof of Lindemann-Weierstrass

Let $\alpha_1, \dots, \alpha_n$ be distinct algebraic numbers. Suppose there exist $b_1, \dots, b_n$ not all zero, such that

$$b_1 e^{\alpha_1} + \dots + b_n e^{\alpha_n} = 0.$$

Let us assume $b_i \neq 0$ for all i.

- Define
$$F(z) = b_1 e^{\alpha_1 z} + \dots + b_n e^{\alpha_n z}.$$
- Then $F(z)$ is an E-function with $F(1) = 0$. Hence the minimal differential equation for $F(z)$ has a singular point at $z = 1$.
- The minimal equation is given by
$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n) F = 0$$
which has no singularities.
- We have a contradiction.

Thank you!