Introduction to E-functions

Frits Beukers

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A general transcendence problem

Let $f(z) \in \overline{\mathbb{Q}}[[z]]$ be power series in $z$ with coefficients in $\overline{\mathbb{Q}}$, with positive radius of convergence $\rho$. We assume $f(z)$ is not algebraic over $\overline{\mathbb{Q}}(z)$.

**Question**

Let $\alpha \in \overline{\mathbb{Q}}$ and suppose $0 < |\alpha| < \rho$. Is $f(\alpha)$ transcendental?
A bizarre function

There exist non-algebraic $f \in \mathbb{Q}[[z]]$ with $\rho = \infty$ such that

$$f(\alpha) \in \overline{\mathbb{Q}} \text{ for all } \alpha \in \overline{\mathbb{Q}}$$

Idea of construction:
Enumerate the elements of $\mathbb{Z}[z]$ by $P_1, P_2, \ldots$ and consider

$$f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k P_1(z) \cdots P_k(z)$$

where $c_k \in \mathbb{Q}$ are chosen such that the resulting $f$ has infinite radius of convergence.

Most of the following (and much more!) can be found in Tanguy Rivoal’s survol

https://rivoal.perso.math.cnrs.fr/articles/EGxups.pdf
Lindemann-Weierstrass theorem

Around 1882 F. Lindemann proved the transcendence of $\pi$. In fact his method yielded more.

**Theorem (Lindemann-Weierstrass)**

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct algebraic numbers. Then

\[ e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n} \]

are linearly independent over $\mathbb{Q}$.

Application: $\pi$ is transcendental.

**Proof:** Suppose $\pi$ were algebraic. Take $\alpha_1 = 0, \alpha_2 = \pi i$. Then Lindemann-Weierstrass implies that $1, e^{\pi i}$ are $\mathbb{Q}$-linear independent, contradicting $e^{\pi i} = -1$. 
**E-function definition**

**Definition**

An entire function $f(z)$ given by a power series

$$
\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k
$$

with $a_k \in \overline{\mathbb{Q}}$ for all $k$, is called an E-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}(z)}$.
2. The height $H(a_0, a_1, \ldots, a_k)$ is bounded by an exponential bound of the form $C^k$, where $C > 0$ depends only on $f$.

**Remark:** Siegel formulated $H(a_1, \ldots, a_k) = O_\epsilon(k!^\epsilon)$ for all $\epsilon > 0$ in his definition. We speak of E-functions in the *broad sense* in that case.
E-function examples

\[ \exp(az) = \sum_{k=0}^{\infty} \frac{a^k z^k}{k!}, \quad a \in \mathbb{Q}^\times \]

\[ J_0(-z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} = \sum_{k \geq 0} \binom{2k}{k} \frac{z^{2k}}{(2k)!} \]

\[ P(z) \in \overline{\mathbb{Q}}[z] \quad (\text{trivial case}) \]

The corresponding differential equations read

\[ y' - ay = 0 \]
\[ zy''' + y' - 4zy = 0 \]
\[ P(z)y' - P'(z)y = 0 \]
Hypergeometric example

A very general example, the confluent hypergeometric series, 

\[ pF_q \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \middle| z^{q+1-p} \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k k!} z^{(q+1-p)k} \]

where \( q \geq p \) (confluence) and \( \alpha_i, \beta_j \in \mathbb{Q} \) for all \( i, j \). 

\( (x)_n \) is the Pochhammer symbol defined by \( x(x+1) \cdots (x+n-1) \).

\( pF_q \) satisfies a linear differential equation of order \( q + 1 \).
Differential ring structure

The E-functions form a so-called differential ring. More precisely,

**Proposition**

Let \( f(z), g(z) \) be E-functions. Then the following functions are again E-functions

- \( f'(z) \)
- \( f(z) + g(z) \)
- \( f(z)g(z) \)

**Theorem (Y. André)**

The units in the ring of E-functions are given by \( \beta e^{\alpha z} \) with \( \alpha, \beta \in \overline{\mathbb{Q}}, \beta \neq 0 \).
Let $L$ be any algebraic number field. An $n \times n$-system of first order linear differential equations over $L$ is given by

$$
\frac{d}{dz} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
$$

where $A_{ij} \in L(z)$ for all $i, j$.

We abbreviate by

$$y' = Ay$$

where $A$ is the $n \times n$-matrix with entries $A_{ij}$.

Let $T(z)$ be the common denominator of the $A_{ij}$. The zeros of $T(z)$ are called the *singularities* of the system.
From equations to systems

Consider the linear $n$-th order differential equation

$$y^{(n)} + p_1y^{(n-1)} + p_2y^{(n-2)} + \cdots + p_{n-1}y' + p_ny = 0, \quad p_i \in L(z)$$

Put

$$y_1 = y, \quad y_2 = y', \quad \ldots, \quad y_n = y^{(n-1)}$$

Note that

$$y'_1 = y_2, \quad y'_2 = y_3, \quad \ldots, \quad y'_{n-1} = y_n.$$ 

Finally,

$$y'_n = -p_1y_n - p_2y_{n-1} - \cdots - p_ny_1.$$ 

Rewrite as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$
Siegel-Shidlovskii theorem

Siegel-Shidlovskii, 1929, 1956

Let \((f_1(z), \ldots, f_n(z))^t\) be a solution vector of a system of first order equations of the form

\[ y'(z) = A(z)y(z) \]

and suppose that the \(f_i(z)\) are E-functions. Let \(T(z)\) be the common denominator of the entries of \(A(z)\). Let \(\alpha \in \overline{\mathbb{Q}}\) and suppose \(\alpha T(\alpha) \neq 0\). Then

\[ \degtr_{\overline{\mathbb{Q}}}(f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)) = \degtr_{\mathbb{C}(z)}(f_1(z), f_2(z), \ldots, f_n(z)) \]

In particular, if the \(f_i(z)\) are algebraically independent over \(\mathbb{C}(z)\) then the values at \(z = \alpha\) are algebraically independent over \(\overline{\mathbb{Q}}\) (or \(\mathbb{Q}\), which amounts to the same).
Algebraic relations between E-function

In the 1960’s and 70’s much energy has gone into showing algebraic independence of (mainly hypergeometric) E-functions. In the 1980’s the tool of differential galois theory was used, which clarified very much of the earlier work.

Example of a relation:
Let $r \in \mathbb{Z}_{>1}$ be odd and consider

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^r} z^k.$$

Then $f(z^r)$ is an $E$-function satisfying a differential equation of order $r$.

A bilinear relation,

$$\sum_{i=0}^{r-1} (-1)^i \times \theta^i f(z) \times \theta^{r-1-i} f(-z) = 0,$$

where $\theta = z \frac{d}{dz}$. 

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Another relation

Let \( r \in \mathbb{Z}_{\geq 1} \) be odd. Define

\[
f(z) = \sum_{k=0}^{\infty} \frac{(1/2)_k}{(k!)^r} z^k.
\]

Then \( f(z^{r^{-1}}) \) is an \( E \)-function satisfying a differential equation of order \( r \). The differential galois group has the form \( C.SO(r, \mathbb{C}) \).

A quadratic relation,

\[
\sum_{i=0}^{r-1} (-1)^i \times \theta^i f(z) \times \theta^{r-1-i} f(z) = z f(z)^2,
\]

where \( \theta = z \frac{d}{dz} \).
Exceptional Galois groups

Consider

\[ f(z) = \sum_{k \geq 0} \frac{(1/14)^k}{(7k)!} z^k. \]

Solution of 7th order differential equation. Katz showed that its Galois group equals \( G_2 \times \mathbb{Z}/7\mathbb{Z}. \)

Some further special Galois groups for suitable parameters (Katz),

- \( q = 8, p = 2, G = C.SL(3) \) (adjoint representation)
- \( q = 8, p = 2, G = C.(SL(2) \times SL(2) \times SL(2)) \)
- \( q = 8, p = 2, G = C.(SL(2) \times Sp(4)) \)
- \( q = 8, p = 2, G = C.(SL(2) \times SL(4)) \)
- \( q = 9, p = 3, G = C.(SL(3) \times SL(3)) \)
Let $f_1, f_2, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations. Then there exists a finite set $S \subset \overline{\mathbb{Q}}$ such that for any algebraic number $\alpha$ not in $S$, polynomial relations over $\overline{\mathbb{Q}}$ between the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ arise from specialization of polynomial relations over $\overline{\mathbb{Q}}(z)$ of the same degree between the functions $f_1(z), \ldots, f_n(z)$.

Proof uses the Siegel-Shidlovskii method.

Theorem (FB (2006), Y. André (2014))

For the exceptional set $S$ one can take the zero set of $zT(z)$.

**Remark**: André’s 2014 proof also holds for E-functions in the broad sense and can be extended to discrete analogues of Siegel-Shidlovskii.
Application to linear (in)dependence

Corollary

Let $f_1, f_2, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations with singularities given by $T(z) = 0$. Let $\alpha \in \overline{\mathbb{Q}}$ with $\alpha T(\alpha) \neq 0$. Then any linear relation over $\overline{\mathbb{Q}}$ between the values $f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)$ arises from specialization of a linear relation over $\overline{\mathbb{Q}}(z)$ between the functions $f_1(z), \ldots, f_n(z)$ (former question of Lang).

Corollary

In particular, $\overline{\mathbb{Q}}(z)$-linear independence of the $f_i(z)$ implies $\overline{\mathbb{Q}}$-linear independence of the $f_i(\alpha)$ when $\alpha T(\alpha) \neq 0$. 
Theorem (FB 2006)

Let $f_1, \ldots, f_n$ be an E-function solution of a first order system of linear differential equations with singularities given by $T(z) = 0$. Suppose they are $\overline{\mathbb{Q}}(z)$-linear independent. Then there exist $E$-functions $e_1(z), \ldots, e_n(z)$ and an $n \times n$-matrix $M(z)$ with entries in $\overline{\mathbb{Q}}[z]$ such that

$$
\begin{pmatrix}
  f_1(z) \\
  \vdots \\
  f_n(z)
\end{pmatrix} = M 
\begin{pmatrix}
  e_1(z) \\
  \vdots \\
  e_n(z)
\end{pmatrix}
$$

and where $(e_1(z), \ldots, e_n(z))$ is vector solution of a system of $n$ homogeneous first order equations with coefficients in $\overline{\mathbb{Q}}[z, 1/z]$. 

Dependence relations are 'trivial'

Corollary
Suppose $\alpha \in \overline{\mathbb{Q}}^\times$ and $b_1, \ldots, b_n \in \overline{\mathbb{Q}}$. Then

$$b_1 f_1(\alpha) + b_2 f_2(\alpha) + \cdots + b_n f_n(\alpha) = 0$$

if and only if $(b_1, b_2, \ldots, b_n)M(\alpha) = 0$.

Proof: Suppose $b_1 f_1(\alpha) + \cdots + b_n f_n(\alpha) = 0$. Then it follows from the theorem that

$$0 = (b_1, \ldots, b_n)M(\alpha) \begin{pmatrix} e_1(\alpha) \\ \vdots \\ e_n(\alpha) \end{pmatrix}.$$ 

Since $e_1(\alpha), \ldots, e_n(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent our corollary follows.
Remarks on linear differential equations

Consider a linear differential equation

\[ q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0 \]

where \( q_i(z) \in \mathbb{C}[z] \) for all \( i \).
The zeros of \( q_n(z) \) are called the *singularities* of the equation, all other points are called *non-singular*.

**Theorem, Cauchy**

Suppose \( a \in \mathbb{C} \) is a non-singular point. Then the solutions of the equation in \( \mathbb{C}[[z - a]] \) form an \( n \)-dimensional \( \mathbb{C} \)-vector space. Furthermore there is an isomorphism of this space with \( \mathbb{C}^n \) given by

\[ y(z) \mapsto (y(a), y'(a), y''(a), \ldots, y^{(n-1)}(a)). \]

Finally, the solutions in \( \mathbb{C}[[z - a]] \) all have positive radius of convergence.
Apparent singular points

- It may happen that there exists a basis of solutions in $\mathbb{C}[[z-a]]$ but $a$ is a singularity. In that case we call $a$ an *apparent singularity*.

- For example, if all solutions around $z=a$ have a zero $a$ then $a$ is an apparent singularity.

In particular $y \mapsto (y(a), y'(a), \ldots, y^{(n-1)}(a))$ is not bijective any more.
We abbreviate our equation
\[ q_n y^{(n)} + q_{n-1} y^{(n-1)} + \cdots + q_1 y' + q_0 y = 0 \]
with \( q_i(z) \in \mathbb{C}(z) \) by
\[ Ly = 0 \]
where \( L \in \mathbb{C}(z)[d/dz] \) denotes the corresponding linear differential operator.

Let \( f \) be a function which satisfies a linear differential equation with coefficients in \( \mathbb{C}(z) \). A minimal differential equation for \( f \) is an equation of smallest possible order satisfied by \( f \).

**Proposition**

Let \( Ly = 0 \) be a minimal differential equation for \( f \). Then for any differential equation \( L_1y = 0 \) satisfied by \( f \) there exists a differential operator \( L_2 \) such that \( L_1 = L_2 \circ L \).
A miraculous theorem

Theorem, Y. André (2000)

Let \( f(z) \) be an E-function. Then \( f(z) \) satisfies a differential equation of the form

\[
z^m y^{(m)} + \sum_{k=0}^{m-1} q_k(z)y^{(k)} = 0
\]

where \( q_k(z) \in \mathbb{Q}[z] \) for all \( k \).

- The equation from André’s theorem need not be the minimal equation of \( f(z) \).
- For example, the function \( (z-1)e^z \) is an E-function, and its minimal differential equation reads \( (z-1)f' = zf \). So we have a singularity at \( z = 1 \). The equation referred to in André’s theorem might be \( f'' - 2f' + f = 0 \).
Corollary, Y. André 2000

Let $f$ be an E-function with rational coefficients. Suppose that $f(1) = 0$. Then the minimal differential equation of $f$ has an apparent singularity at $z = 1$.

The simplest example is again $f = (z - 1)e^z$, an E-function which vanishes at $z = 1$. Its minimal differential equation is $(z - 1)f' = zf$. 
Proof. Consider $f(z)/(1 - z)$. We will show that it is an E-function again. It is certainly an entire analytic function. Suppose that

$$f(z) = \sum_{r \geq 0} \frac{f_r}{r!} z^r, \quad f_r \in \mathbb{Q}$$

Then the power series of $f(z)/(1 - z)$ reads

$$\frac{f(z)}{1 - z} = \sum_{r \geq 0} \frac{g_r}{r!} z^r$$

where

$$g_r = r! \sum_{k=0}^{r} \frac{f_k}{k!}.$$ 

Suppose that the common denominator of $f_0, \ldots, f_r$ and the sizes $|f_r|$ are bounded by $C^r$ for some $C > 0$. Then clearly the common denominators of $g_0, \ldots, g_r$ are again bounded by $C^r$. 

Proof of André’s corollary
Recall

\[ g_r = r! \sum_{k=0}^{r} \frac{f_k}{k!}. \]

To estimate the size of \(|g_r|\) we use the fact that

\[ 0 = f(1) = \sum_{k \geq 0} \frac{f_k}{k!}. \]

More precisely,

\[
|g_r| = \left| -r! \sum_{k>r} \frac{f_k}{k!} \right|
\leq \sum_{k>r} |f_k|/(k-r)!
\leq \sum_{k>r} C^k/(k-r)! < C^r e^C
\]

So \(|g_r|\) is exponentially bounded in \(r\). Hence \(f(z)/(1-z)\) is an E-function.
Notice that this argument only works if \( f(z) \) is an E-function with *rational coefficients*, i.e. in \( \mathbb{Q} \).

By André’s theorem \( f(z)/(1 - z) \) satisfies a differential equation without singularity at \( z = 1 \).

Hence its minimal differential equation has a full solution space \( V \) of analytic solutions at \( z = 1 \).

The solutions of the minimal equation of \( f(z) \) can be found by multiplying the elements from \( V \) by \( z - 1 \).

This means that the minimal equation for \( f(z) \) has a full space of analytic solutions all vanishing at \( z = 1 \).

So \( z = 1 \) is apparent singularity.
Generalizing André’s corollary

Using a combination of André’s theorem and some differential galois theory one can prove the following result.

**Theorem, FB (2006)**

Let \( f(z) \) be an \( E \)-function with coefficients in \( \overline{\mathbb{Q}} \). Suppose that \( f(1) = 0 \). Then 1 is an apparent singularity of the minimal differential equation satisfied by \( f \).
Another proof of Lindemann-Weierstrass

Let \( \alpha_1, \ldots, \alpha_n \) be distinct algebraic numbers. Suppose there exist \( b_1, \ldots, b_n \) not all zero, such that

\[
b_1 e^{\alpha_1} + \cdots + b_n e^{\alpha_n} = 0.
\]

Let us assume \( b_i \neq 0 \) for all \( i \).

- Define
  
  \[
  F(z) = b_1 e^{\alpha_1 z} + \cdots + b_n e^{\alpha_n z}.
  \]

- Then \( F(z) \) is an E-function with \( F(1) = 0 \). Hence the minimal differential equation for \( F(z) \) has a singular point at \( z = 1 \).

- The minimal equation is given by
  
  \[
  (D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)F = 0
  \]

  which has no singularities.

- We have a contradiction.
Thank you!