# Viewpoints on the $p$-curvature

Xavier Caruso

University of Bordeaux

**E**-functions, **G**-functions and **P**eriods

January 10th, 2023

# Viewpoints on linear differential equations

$$a_r(x) \cdot y^{(r)} + a_{r-1}(x) \cdot y^{(r-1)} + \cdots + a_1(x) \cdot y' + a_0(x) \cdot y = 0$$

**Differential operators**
$L = a_r \partial^r + \cdots + a_1 \partial + a_0 \in k(x)\langle \partial \rangle$
$L(y) = 0$

**Differential systems**
$Y \in k(x)^r$, $A \in M_r(k(x))$
$Y' + AY = 0$

**Modules with connections**
$M$ vector space over $k(x)$
$\partial : M \to M$ such that
$\partial(fm) = f'm + f\partial(m)$

## Some examples

☞ $y' - y = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad y = e^x$

☞ $x \cdot y' - n \cdot y = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad y = x^n$

☞ $x(x-1) \cdot y'' + (c - (a+b+1)x) \cdot y' - ab \cdot y = 0 \qquad \displaystyle\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$

☞ $(x\partial - \alpha_1) \cdots (x\partial - \alpha_n) - x(x\partial - \beta_1) \cdots (x\partial - \beta_n)$

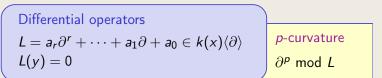$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_n)_n}{(\beta_1 + 1)_n \cdots (\beta_n + 1)_n} x^n$$

**Theorem** (Beukers-Heckmann)

The above differential equation has a basis of algebraic solutions iff:

☞ the $\alpha_i$ and the $\beta_j$ are $2n$ distinct elements of $\mathbb{Q}/\mathbb{Z}$,

☞ for all $r$, the sets $A_r = \{r\alpha_1, \ldots, r\alpha_n\}$ and
$B_r = \{r\beta_1, \ldots, r\beta_n\}$ are intertwined in $\mathbb{Q}/\mathbb{Z}$.

# Viewpoints on linear differential equations

$$a_r(x) \cdot y^{(r)} + a_{r-1}(x) \cdot y^{(r-1)} + \cdots + a_1(x) \cdot y' + a_0(x) \cdot y = 0$$

Differential operators
$L = a_r \partial^r + \cdots + a_1 \partial + a_0 \in k(x)\langle \partial \rangle$
$L(y) = 0$

*p*-curvature
$\partial^p \bmod L$

Differential systems
$Y \in k(x)^r$, $A \in M_r(k(x))$
$Y' + AY = 0$

*p*-curvature
$A_p$ avec :
$A_1 = A$
$A_{i+1} = A_i' + AA_i$

Modules with connections
$M$ vector space over $k(x)$
$\partial : M \to M$ such that
$\partial(fm) = f'm + f\partial(m)$

*p*-curvature
$\partial^p : M \to M$
$\partial^p(fm) = f^{(p)}m + f\partial^p(x) = f\partial^p(m)$

# *p*-curvature and solutions

**Theorem** (Cartier)

dimension of the spaces of solutions
*over the subfield of constants*

= dimension of the kernel of the *p*-curvature
*over $k(x)$*

# *p*-curvature and solutions

**Theorem** (Cartier)

over the subfield of constants
dimension of the spaces of rational solutions

= dimension of the kernel of the *p*-curvature
over $k(x)$

# $p$-curvature and solutions

**Theorem** (Cartier)

dimension <u>over the subfield of constants</u> of the spaces of algebraic solutions

$=$ dimension of the kernel of the $p$-curvature <u>over $k(x)$</u>

# *p*-curvature and solutions

**Theorem** (Cartier)

dimension of the spaces of series solutions
*over the subfield of constants*

= dimension of the kernel of the *p*-curvature
*over $k(x)$*

# $p$-curvature and solutions

**Theorem** (Cartier)

dimension of the spaces of series solutions
$\overbrace{\phantom{xxxxxxxxxxxx}}$ over the subfield of constants

$=$ dimension of the kernel of the $p$-curvature
$\underbrace{\phantom{xxxxxx}}$ over $k(x)$

$(M, \partial)$ module with connection
$\dim_{k(x^p)} \ker \partial = \dim_{k(x)} \ker \partial^p$
$\ker \partial^p = k(x) \otimes_{k(x^p)} \ker \partial$

$Y' + AY = 0$ with $Y = Y_0 + Y_1 x + Y_2 x^2 + \cdots$
*Recurrence*: $nY_n = f_n(Y_0, \ldots, Y_{n-1})$

Necessary condition to the existence of solutions:
$f_p(Y_0, \ldots, Y_{p-1}) = 0$

# Grothendieck conjecture

Let $L \in \mathbb{Q}(x)\langle \partial \rangle$.

    The differential equation $L(y) = 0$
has a basis of algebraic solutions

**iff** the $p$-curvature of $L$ mod $p$
vanishes for almost all prime $p$

    $\partial^p \equiv 0 \pmod{L, p}$

> **Katz Conjecture**
>
> The $p$-curvatures
> are dense in
> the Lie algebra
> of the Galois group

Let $L \in \mathbb{Q}[X]$ separable.

    The polynomial $L$ splits over $\mathbb{Q}$

**iff** the polynomial $L$ mod $p$
splits over $\mathbb{F}_p$ for almost all prime $p$

    $X^p \equiv X \pmod{L, p}$

> **Chebotarev**
>
> Every element of
> the Galois group
> is a Frobenius
> at some $p$

# Naive computation of the *p*-curvature

> **Reminder**
>
> The *p*-curvature of $Y' = AY$ is $A_p$ with
>
> $A_1 = A$
>
> $A_{i+1} = A_i' + AA_i$

**First order**

The *p*-curvature of $y' = a(x)y$ is

$$-\frac{d^{p-1}a(x)}{d^{p-1}x} - a(x)^p$$

**In general**

The recurrence gives an algorithm with complexity $\tilde{O}(dr^\omega p^2)$

To be compared with the size of $A_p$ which is $O(dr^2 p)$

# Fast computation of the $p$-curvature

$Y' + AY = 0$ with $Y = Y_0 + Y_1 x + Y_2 x^2 + \cdots$

*Recurrence*: $n Y_n = f_n(Y_0, \ldots, Y_{n-1})$

Necessary condition to the existence of solutions:

$f_p(Y_0, \ldots, Y_{p-1}) = 0$

$$k[[x]]^{\mathrm{dp}} = \left\{ f(x) = \sum_{i=0}^{\infty} c_i \frac{x^i}{i!}, \ c_i \in k \right\}$$

$x^p = 0$ in $k[[x]]^{\mathrm{dp}}$

**Proposition**

Let $A \in M_r(k(x))$ with no pole at 0.

The differential system $Y' = AY$ admits

a fundamental system of solutions over $k[[x]]^{\mathrm{dp}}$

# Fast computation of the *p*-curvature (cont.)

### Observation

Let $A \in M_r(k(x))$ with no pole at 0.

For all solution $Y = Y_0 + Y_1 x + Y_2 \frac{x^2}{2!} + \cdots + Y_p \frac{x^p}{p!} + \cdots$
of $Y' = AY$,

we have $A_p Y_0 = -Y_p$ in $k[x]/x^p$

### Corollary (Bostan, C., Schost)

There exists an algorithm that compute the *p*-curvature
in complexity $\tilde{O}(dr^\omega p)$

There exists an algorithm that compute the
the similarity invariant of the *p*-curvature
in complexity $\tilde{O}(d^{\omega + \frac{3}{2}} r^{\omega + 1} \sqrt{p})$

### Method

Use the theorem to compute $A_p$ mod $(x - a)^p$ for enough values of $a$

# The characteristic polynomial of the $p$-curvature

We introduce the *Euler operator* $\theta = x\partial$

$$
\begin{aligned}
k[x]\langle\partial\rangle &\longrightarrow k[\theta]\langle x^{\pm 1}\rangle && \left[\, x\theta = (\theta-1)x \,\right] \\
\partial &\longmapsto x^{-1}\theta \\
L &\longmapsto R
\end{aligned}
$$

$$
\begin{aligned}
A \in M_r\big(k(x)\big) && B \in M_s\big(k(\theta)\big) \\
\rightsquigarrow A_p && \rightsquigarrow B_p = B(\theta)\,B(\theta+1)\cdots B(\theta+p-1)
\end{aligned}
$$

**Renormalized characteristic polynomials**

$$
P(T, x^p) = \det(T - A_p) \cdot a_r(x)^p
$$
$$
Q(T, \theta^p - \theta) = \det(T - B_p) \cdot b_s(\theta)\, b_s(\theta+1) \cdots b_s(\theta+p-1)
$$

**Theorem** (Bostan, C., Schost)

We have $P(T, U) = Q(U, TU)$

There exists an algorithm that computes $\det(T - A_p)$
in complexity $\tilde{O}\big(d\,(d+r)^\omega \sqrt{p}\big)$

## Example: hypergeometric operators

$$\mathcal{H}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$$
$$= (x\partial - \alpha_1) \cdots (x\partial - \alpha_n) - x(x\partial - \beta_1) \cdots (x\partial - \beta_n)$$
$$= (\theta - \alpha_1) \cdots (\theta - \alpha_n) - x \cdot (\theta - \beta_1) \cdots (\theta - \beta_n)$$

More generally, consider $\mathcal{H}(\alpha; \beta) = \alpha(\theta) - x \cdot \beta(\theta)$

$$B = \left( \frac{\alpha(\theta)}{\beta(\theta)} \right) \quad ; \quad B_p = \left( \frac{\alpha(\theta)\,\alpha(\theta+1)\cdots\alpha(\theta+p-1)}{\beta(\theta)\,\beta(\theta+1)\cdots\beta(\theta+p-1)} \right)$$

$$Q(T, \theta^p - \theta)$$
$$= \det(T - B_p) \cdot \beta(\theta)\,\beta(\theta+1) \cdots \beta(\theta+p-1)$$
$$= \beta(\theta)\,\beta(\theta+1) \cdots \beta(\theta+p-1) \cdot T$$
$$\qquad - \alpha(\theta)\,\alpha(\theta+1) \cdots \alpha(\theta+p-1)$$

We retrieve $P(T, x^p) \propto \det(T - A_p)$
after the changes of variables $T \rightsquigarrow x^p$ and $\theta^p - \theta \rightsquigarrow x^p T$

Merci pourbure
votre attention